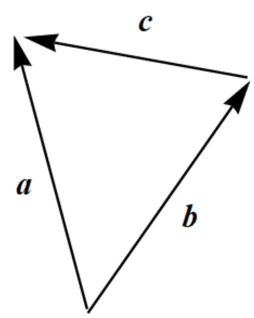
Exercise 27

Using the dot product, prove the converse of the Pythagorean theorem. That is, show that if the lengths of the sides of a triangle satisfy $a^2 + b^2 = c^2$, then the triangle is a right triangle.

Solution

Suppose there's a triangle formed by vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ with magnitudes $\|\mathbf{a}\| = a$, $\|\mathbf{b}\| = b$, and $\|\mathbf{c}\| = \sqrt{a^2 + b^2}$, respectively.



From this figure, $\mathbf{a} = \mathbf{b} + \mathbf{c}$, which means

$$\mathbf{c} = \mathbf{a} - \mathbf{b}.$$

Square both sides.

$$\mathbf{c} \cdot \mathbf{c} = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$$

$$\begin{aligned} \|\mathbf{c}\|^2 &= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\ &= \|\mathbf{a}\|^2 - 2(\mathbf{a} \cdot \mathbf{b}) + \|\mathbf{b}\|^2 \\ &= \|\mathbf{a}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta + \|\mathbf{b}\|^2 \end{aligned}$$

θ represents the angle between $\mathbf{a}$ and $\mathbf{b}$. Substitute the magnitudes on both sides.

$$a^2 + b^2 = a^2 - 2ab\cos\theta + b^2$$

Solve for $\cos\theta$.

$$0 = -2ab\cos\theta$$

$$\cos\theta = 0.$$

Therefore, $\theta = \pi/2$, and the triangle is a right triangle.